Generalized Abel-Jacobi map on Lawson homology

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Abstract

We construct an Abel-Jacobi type map on the homologically trivial part of Lawson homology groups. It generalizes the Abel-Jacobi map constructed by Griffiths. By using a result of H. Clemens, we give some examples of smooth projective manifolds with infinite generated Lawson homology groups $L_pH_{2p+k}(X,\mathbb{Q})$ when $k>0$.

Contents

1 Introduction

In this paper, all varieties are defined over $\mathbb{C}$. Let X be a smooth projective variety with dimension n. Recall that the **Hodge filtration**

$$\cdots \subseteq F^iH^k(X,\mathbb{C}) \subseteq F^{i-1}H^k(X,\mathbb{C}) \subseteq \cdots \subseteq F^0H^k(X,\mathbb{C}) = H^k(X,\mathbb{C})$$

is defined by

$$F^qH^k(X,\mathbb{C}) := \bigoplus_{i\geq q} H^{i,k-i}(X).$$

Note that $F^qH^k(X,\mathbb{C})$ vanishes if $q>k$.

In [G], Griffiths generalized the Jacobian varieties and the Abel-Jacobi map on smooth algebraic curves to higher dimensional smooth projective varieties.

Definition 1.1 *The* ***q-th intermediate Griffiths Jacobian*** *of a smooth projective variety X is defined by*

$$\begin{aligned} J^q(X): &= H^{2q-1}(X,\mathbb{C})/\{F^qH^{2q-1}(X,\mathbb{C})+H^{2q-1}(X,\mathbb{Z})\} \\ &\cong F^{n-q+1}H^{2n-2q+1}(X,\mathbb{C})^*/H^{2q-1}(X,\mathbb{Z})^*. \end{aligned}$$

Let $\mathcal{Z}_p(X)$ be the space of algebraic p-cycles on X. Set $\mathcal{Z}^{n-p}(X)\equiv\mathcal{Z}_p(X)$. There is a natural map

$$cl_q:\mathcal{Z}^q(X)\to H^{2q}(X,\mathbb{Z})$$

called **the cycle class map**. Set

$$\mathcal{Z}_{n-q}(X)_{hom}:=\mathcal{Z}^q(X)_{hom}:=\ker cl_q$$

Definition 1.2 *The* ***Abel-Jacobi map***

$$\Phi^q:\mathcal{Z}^q(X)_{hom}\to J^q(X)$$

sends $\varphi\in\mathcal{Z}^q(X)_{hom}$ to Φ^q_φ, where Φ^q_φ is defined by

$$\Phi^q_\varphi(\omega):=\int_U\omega,\quad \omega\in F^{n-q+1}H^{2n-2q+1}(X,\mathbb{C}).$$

Here $\varphi=\partial U$ and U is an integral current of dimension $2n-2q+1$.

Now let

$$J^{2q-1}(X)_{alg}\subseteq J^{2q-1}(X)$$

be the largest complex subtorus of $J^{2q-1}(X)$ whose tangent space is contained in $H^{q-1,q}(X)$. It can be proved that $\Phi^q(\mathcal{Z}^q(X)_{alg})$ is a subtorus of $J^{2q-1}(X)$ contained in $J^{2q-1}(X)_{alg}$ (cf. [V1], Corollary 12.19), where $\mathcal{Z}^q(X)_{alg}\subseteq\mathcal{Z}^q(X)$ are the subset of codimension q-cycles which are algebraically equivalent to zero.

The **Griffiths group** of codimension q-cycles is defined to

$$\mathrm{Griff}^q(X):=\mathcal{Z}^q(X)_{hom}/\mathcal{Z}^q(X)_{alg}$$

Therefore we can define the transcendental part of the Abel-Jacobi map

$$\Phi^q_{tr}:\mathrm{Griff}^q(X)\to J^q(X)_{tr}:=J^{2q-1}(X)/J^{2q-1}(X)_{alg} \tag{1}$$

as the factorization of Φ^q.

By using this, Griffiths showed the following:

Theorem 1.1 ([G]) *Let $X \subset \mathrm{P}^4$ be a general quintic threefold, the Griffiths group* $\mathrm{Griff}^2(X)$ *is nontrivial, even modulo torsion.*

Remark 1.1 *Clemens has obtained further results: Under the same assumption as those in Theorem 1.1,* $\mathrm{Griff}^2(X) \otimes \mathbb{Q}$ *is an infinitely generated $\mathbb{Q}$-vector space [Cl].*

In this paper, the Griffiths' Abel-Jacobi map is generalized to the spaces of the homologically trivial part of Lawson homology groups.

Definition 1.3 *The* ***Lawson homology*** $L_pH_k(X)$ *of p-cycles is defined by*

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad for \quad k \geq 2p \geq 0,$$

where $\mathcal{Z}_p(X)$ is provided with a natural topology (cf. [F1], [L1]). For general background, the reader is referred to [L2].

In [FM], Friedlander and Mazur showed that there are natural maps, called **cycle class maps**

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X).$$

Define

$$L_pH_k(X)_{hom} := \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}$$

and

$$L_pH_k(X, \mathbb{Q}) := L_pH_k(X) \otimes \mathbb{Q}.$$

The domain of Abel-Jacobi map can be reduced to Griffiths groups as in (1). Similarly, our generalized Abel-Jacobi map is defined on homologically trivial part of Lawson homology groups. As an application, we show that the non-triviality of certain Lawson homology group.

The main result in this paper is the following:

Theorem 1.2 *Let X be a smooth projective variety. There is a well-defined map*

$$\Phi : L_pH_{2p+k}(X)_{hom} \longrightarrow \left\{ \bigoplus_{r>k+1, r+s=k+1} H^{p+r,p+s}(X) \right\}^* \Big/ H_{2p+k+1}(X, \mathbb{Z})$$

which generalizes Griffiths' Abel-Jacobi map defined in [G]. Moreover, for any $p > 0$ and $k \geq 0$, we find examples of projective manifolds X for which the image of the map on $L_pH_{k+2p}(X)_{hom}$ is infinitely generated.

As the application of the main result together Clemens' Theorem (Remark 1.1), we obtain

Theorem 1.3 *For any $k \geq 0$, there exist a projective manifold X of dimension $k+3$ such that $L_1H_{k+2}(X)_{hom} \otimes \mathbb{Q}$ is nontrivial, in fact, infinite dimensional over $\mathbb{Q}$.*

Using the Projective Bundle Theorem proved by Friedlander and Gabber in [FG], we have the following result:

Theorem 1.4 *For any $p > 0$ and $k \geq 0$, there exist a smooth projective variety X such that $L_pH_{k+2p}(X)_{hom} \otimes \mathbb{Q}$ is an infinite dimensional vector space over $\mathbb{Q}$.*

In section 2, we will review the minimal background materials about Lawson homology and point out its relation to Griffiths groups. In section 3, we give the definition the generalized Abel-Jacobi map. In section 4, the non-triviality of the generalized Abel-Jacobi map is proved by using Griffiths and Clemens' results through examples. The construction in our examples also shows this generalized Abel-Jacobi map really generalizes Griffiths' result in [G].

2 Lawson homology

Let X be a projective variety of dimension n. Denote by $\mathcal{C}_p(X)$ the space of effective algebraic p-cycles on X and by $\mathcal{Z}_p(X)$ the space of algebraic p-cycles on X. There is a natural, compactly generated topology on $\mathcal{C}_p(X)$ (resp. $\mathcal{Z}_p(X)$) and therefore $\mathcal{C}_p(X)$ (resp. $\mathcal{Z}_p(X)$) carries a structure of an abelian topological group ([F1], [L1]).

The **Lawson homology** $L_pH_k(X)$ of p-cycles is defined by

$$L_pH_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X)) \quad for \quad k \geq 2p \geq 0.$$

It has been proved by Friedlander in [F1] that

$$L_pH_k(X) \cong \varinjlim \pi_k(\mathcal{C}_p(X)_\alpha)$$

for all $k > 0$, where the limit is taken over the connected components of $\mathcal{C}_p(X)$ with respect to the action of $\pi_0(\mathcal{Z}_p(X))$. For a detailed discussion of this construction and its properties we refer the reader to [FM], §2 and [FL1],§1.

In [FM], Friedlander and Mazur showed that there are natural maps, called **cycle class maps**

$$\Phi_{p,k} : L_pH_k(X) \to H_k(X)$$

where $H_k(X)$ is the singular homology with the integral coefficient.

Define $L_pH_k(X)_{hom}$ to be the **homologically trivial part of Lawson homology group** $L_pH_k(X)_{hom}$, i.e.,

$$L_pH_k(X)_{hom} := \ker\{\Phi_{p,k} : L_pH_k(X) \to H_k(X)\}.$$

It was proved by Friedlander [F1] that $L_pH_{2p}(X) \cong \mathcal{Z}_p(X)/\mathcal{Z}_p(X)_{alg}$. Therefore we have

$$L_pH_{2p}(X)_{hom} \cong \mathrm{Griff}_p(X), \tag{2}$$

where $\mathrm{Griff}_p(X) := \mathrm{Griff}^{n-p}(X)$.

For general background on Lawson homology, the reader is referred to [L2].

3 The definition of generalized Abel-Jacobi map on $L_pH_{2p+k}(X)_{hom}$

In this section, X denotes a smooth projective algebraic manifold with dimension n. Now $\mathcal{Z}_p(X)$ is an abelian topological group with an identity element, the "null" p-cycle.

For $[\varphi] \in L_pH_{2p+k}(X)$, we can construct an integral $(2p+k)$-cycle c in X with $\Phi_{p,2p+k}([\varphi]) = [c]$, where $[c]$ is the homology class of c.

To see how to construct c from $[\varphi]$ for the case that $p = 0$, the reader is referred to [FL2].

We will use this construction several times in the following. We briefly review the construction here.

A class

$$[\varphi] \in L_pH_{2p+k}(X) = \varinjlim \pi_k(\mathcal{C}_p(X))$$

is represented by a map

$$\varphi : S^k \to \mathcal{C}_p(X).$$

(For $k = 0$, $[\varphi]$ is represented by a difference of such maps.)

We may assume φ to be piecewise linear (PL for short) with respect to a triangulation of $\mathcal{C}_p(X) \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ respecting the smooth stratified structure ([Hi]). Here Γ_i is a subcomplex for every $i > 0$.

Let φ be as above and fix $s_0 \in S^k$ and $x_0 \in \mathrm{Supp}\ (\varphi(s_0)) \subset X$. There exist affine coordinates $(z_1, \cdots z_p, \zeta_1, \cdots, \zeta_{n-p})$ on X with $x_0 = 0$ such that the projection $pr_1(z, \zeta) = z$, when restricted to $U \times U' = \{(z, w) : |z| < 1 \ and\ |w| < 1\}$, gives a proper (hence finite) map $pr : \mathrm{Supp}\ (\varphi(s_0)) \cap (U \times U') \to U$. Slicing this cycle $\varphi(s_0)|_{U\times U'}$ by this projection gives a PL map $\sigma : U \to SP^d(U')$ (with respect to a triangulation of $SP^d(U')$) for some d. Furthermore, given any such a map, we can construct a cycle in $U \times U'$. (cf. [FL2].) Choose a finite number of such product neighborhood $U_\alpha \times U'_\alpha$, $\alpha = 1, \cdots, K$, so that the union of $U_\alpha \times U'_\alpha(\frac{1}{2})$ covers $\mathrm{Supp}\ (\varphi(s_0)$. After shrinking each U'_α slightly, we can find a neighborhood $\mathcal{N}$ of s_0 in S^k such that $pr : \mathrm{Supp}\ (\varphi(s)) \cap (U_\alpha \times U'_\alpha) \to U_\alpha$ for all $s \in \mathcal{N}$ and for all α. Then φ is PL in $\mathcal{N}$ if and only if $\sigma : \mathcal{N} \times U_\alpha \to SP^d(U'_\alpha)$ is PL for all α. One defines the cycle $c(\varphi)$ in each neighborhood $\mathcal{N} \times U_\alpha \times U'_\alpha$ by graphing this extended

σ. From the construction, the cycle $c(\varphi)$ depends only on the PL map φ. (The argument here is from [[FL1], page 370-371].)

Lemma 3.1 *The homology class $c_\varphi := (pr_2)_*(c(\varphi))$ is independent of the choice of PL map $\varphi : S^k \to \mathcal{C}_p(X)$ in $[\varphi]$, where $pr_2 : S^k \times X \to X$ is the projection onto the second factor.*

Proof. Suppose that $\varphi' : S^k \to \mathcal{C}_p(X)$ is another PL map in $[\varphi]$. Hence, we have a continuous map $H : S^k \times [0,1] \to \mathcal{C}_p(X)$ such that $H|_{S^k\times\{0\}} = \varphi$ and $H|_{S^k\times\{1\}} = \varphi'$. Furthermore, this map can be chosen to be PL with respect to the triangulation of $\mathcal{C}_p(X)$. Therefore, by the same construction as above, we obtain that an integral current $c_H := (pr_2)_*(c(H))$. It is clear $\partial(c_H) = c_\varphi - c_{\varphi'}$ since the push-forward $(pr_2)_*$ commutes with the boundary map ∂.

□

Alternatively, the restriction of φ to the interior of each top dimensional simplicies Δ_j^k ($1 \le j \le N$, N is the number of top dimensional simplices) gives a map $\varphi : \Delta_j^k \to \Gamma_{n_j}$, where Δ_j^k is the j-th k-dimensional simplex and n_j is the maximum number such that Γ_{n_j} contains the image of $\varphi|_{\Delta_j^k}$.

The piecewise linear property of φ with respect to the stratified structure of $\mathcal{C}_p(X)$ have the following property:

($*$) *For each $s \in \Delta_j^k$, $\varphi(s) = \sum_i a_i(s)V_i(s)$ with the property that $a_i(s) = a_i$ is constant in s and $V_i(s)$ is irreducible.*

For each j, $1 \le j \le N$, set $Z_{\Delta_j^k} := \sum_i a_i Z_{i,j}^k$, where $Z_{i,j}^k := \{(s,z) \in \Delta_j^k \times X | z \in V_i(s)\}$. It is clear that $Z_{\Delta_j^k}$ is an integral current. Therefore,

$$\phi_j^k := (pr_2)_*(Z_{\Delta_j^k}) \tag{3}$$

is then an integral current of real dimension $2p + k$, where $pr_2 : \Delta_j^k \times X \to X$ is the projection onto the second factor. Set $Z(\varphi) := \sum_{j=1}^N \phi_j^k$.

Lemma 3.2 *The closure of $Z(\varphi)$ is an integral cycle in X.*

Proof. Since φ is piecewise linear with respect to the triangulation of $\mathcal{C}_p(X)$. The image of φ on each $(k-1)$-dimensional simplex Δ_i^{k-1} is in Γ_{m_i}, where m_i is the maximum number such that Γ_{m_i} contains the image of $\varphi|_{\Delta_i^{k-1}}$. Each $\varphi|_{\Delta_i^{k-1}}$ defines a current $\phi_i^{k-1} := (pr_2)_*(Z_{\Delta_i^{k-1}})$ as in (3). The sum

$$\sum_i \phi_i^{k-1}$$

is zero since, for each ϕ_i^{k-1}, there is exactly one $\phi_{i'}^{k-1}$ such that they have the same support but different orientation.

$\square$

Let c_φ be the total $(k+2p)$-cycle in X determined by $[\varphi]$. We will simply use c instead of c_φ unless it arises confusion.

Remark 3.1 *c_φ, as current, has restricted type $c_\varphi = [c_\varphi]_{p+k,p} + [c_\varphi]_{p+k-1,p+1} + \cdots + [c_\varphi]_{p,p+k}$.*

If c is homologous to zero, we denote it by $c \overset{hom}{\sim} 0$, i.e., $[\varphi] \to 0$ in $H_{2p+k}(X,\mathbb{Z})$ under the natural transformation $\Phi_{p,2p+k} : L_pH_{2p+k}(X) \to H_{2p+k}(X,\mathbb{Z})$ (see, e.g., [L2], p.185). This condition translates into the fact that there exists an integral topological $(2p+k+1)$-chain $\tilde{c}$ such that $\partial\tilde{c} = c$.

We denote by $\mathrm{Map}(S^k, \mathcal{C}_p(X))$ the set of piecewise linear maps with respect to a triangulation of $\mathcal{C}_p(X)$ from the k-dimensional sphere to the abelian topological monoid $\mathcal{C}_p(X))$ of p-cycles.

Set

$$\mathrm{Map}(S^k, \mathcal{C}_p(X))_{hom} \subset \mathrm{Map}(S^k, \mathcal{C}_p(X))$$

the subset of such maps $\varphi : S^k \to \mathcal{C}_p(X)$ whose total cycles c_φ is homologous to zero in $H_{2p+k}(X,\mathbb{Z})$. There is a natural induced compact open topology on the space of such maps $\mathrm{Map}(S^k, \mathcal{C}_p(X))$ (see, e.g., Whitehead [W]).

Now $\mathcal{Z}_p(X)$ is the group completion of the topological monoid $\mathcal{C}_p(X)$ (cf. [F1], [L1]). In the following, we will denote by $\mathrm{Map}(S^k, \mathcal{Z}_p(X))$ the set of piecewise linear maps with respect to a triangulation of $\mathcal{Z}_p(X)$ from the k-dimensional sphere to the abelian topological group $\mathcal{Z}_p(X)$ of p-cycles.

Let $\varphi : S^k \to \mathcal{Z}_p(X)$ be a PL map which is homotopic to zero. Hence there exists a map $\tilde{\varphi} : D^{k+1} \to \mathcal{Z}_p(X)$ such that $\tilde{\varphi}$ is PL with respect to a triangulation of $\mathcal{Z}_p(X)$ and $\tilde{\varphi}|_{S^k} = \varphi$. Then $\tilde{\varphi}$ determines an integral current, i.e., the total $(k+1+2p)$-chain $\tilde{c}$ such that the boundary of $\tilde{c}$ is c, i.e., $\partial\tilde{c} = c$. From the definition, we have $\varphi \in \mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom}$. Denote by $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0$ the subspace of $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom}$ consisting of elements φ which are homotopic to zero.

3.1 The generalized Abel-Jacobi map on $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom}$

In this subsection, suppose that

$$c \overset{hom}{\sim} 0, \ \textit{i.e.,}\ [\varphi] \to 0 \in H_{2p+k}(X,\mathbb{Z})$$

under the natural transformation $\Phi_{p,2p+k} : L_pH_{2p+k}(X) \to H_{2p+k}(X,\mathbb{Z})$ (see, e.g., [L2], p.185). This condition translates into the fact that there exists an integral topological $(2p+k+1)$-chain $\tilde{c}$ such that $\partial\tilde{c} = c$.

Consider

$$\omega \in \left\{ \bigoplus_{r\geq k+1, r+s=k+1} \mathcal{E}^{p+r,p+s} \right\}, \quad d\omega = 0$$

and we define

$$\Phi_\varphi(\omega) = \int_{\tilde{c}} \omega.$$

We claim:

Proposition 3.1 *Φ_φ is well-defined, i.e., $\Phi_\varphi(\omega)$, as an element in*

$$\left\{ \bigoplus_{r,s\geq 0, r+s=k+1} H^{p+r,p+s}(X) \right\}^* \Big/ H_{2p+k+1}(X,\mathbb{Z}),$$

depends only on the cohomology class of ω. Here we identify $H_{2p+k+1}(X,\mathbb{Z})$ with the image of the composition

$$H_{2p+k+1}(X,\mathbb{Z}) \xrightarrow{\rho} H_{2p+k+1}(X,\mathbb{C}) \cong H^{2p+k+1}(X,\mathbb{C})^* \xrightarrow{\pi} \left\{ \bigoplus_{r,s\geq 0, r+s=k+1} H^{p+r,p+s}(X) \right\}^*, \tag{4}$$

where ρ is the coefficient homomorphism and π is the projection onto the subspace.

Proof. We need to show

1. For another choice of $\omega' \in \bigoplus_{r\geq k+1, r+s=k+1} \mathcal{E}^{p+r,p+s}$, $\omega - \omega' = d\alpha$, we have $\int_{\tilde{c}} \omega = \int_{\tilde{c}} \omega'$.
2. If $\tilde{c}'$ is another integral topological chain such that $\partial \tilde{c}' = c$, then we also have $\int_{\tilde{c}} \omega = \int_{\tilde{c}'} \omega$, where $\tilde{c}'$ is the currents determined by $\tilde{\varphi}$.

To show the part 1), note that we can choose α such that $\omega - \omega' = d\alpha$ for some α with $\alpha^{r,s} = 0$ if $r \leq k+p$ by the Hodge decomposition theorem for differential forms on X. Hence

$$\int_{\tilde{c}} \omega - \int_{\tilde{c}} \omega' = \int_{\tilde{c}} d\alpha = \int_c \alpha = 0$$

by the Stokes Theorem and the reason of type. This shows that the definition of Φ_φ is independent of the cohomology class of

$$[\omega] \in \left\{ \bigoplus_{r\geq k+1, r+s=k+1} H^{p+r,p+s}(X) \right\}.$$

To show the part 2), note that $\partial(\tilde{c} - \tilde{c}') = 0$ and hence $\tilde{c} - \tilde{c}' = \lambda$ is an integral topological cycle and hence $\int_\lambda$ lies in the image of the composition in (4). Hence $\int_\lambda$ is well-defined independently of the choice of $\tilde{c}$ such that $\partial \tilde{c} = c$, as an element in

$$\left\{\bigoplus_{r,s\geq 0, r+s=k+1} H^{p+r,p+s}(X)\right\}^* \Big/ H_{2p+k+1}(X,\mathbb{Z}).$$

Hence we obtain a well-defined element

$$\Phi_\varphi \in \left\{\bigoplus_{r\geq k+1, r+s=k+1} H^{p+r,p+s}(X)\right\}^* \Big/ H_{2p+k+1}(X,\mathbb{Z}).$$

This completes the proof of the Proposition.

□

Therefore by Proposition 3.1 we have a well-defined homomorphism

$$\Phi : \mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom} \to \left\{\bigoplus_{r\geq k+1, r+s=k+1} H^{p+r,p+s}(X)\right\}^* \Big/ H_{2p+k+1}(X,\mathbb{Z}) \tag{5}$$

given by $\Phi(\varphi) = \Phi_\varphi$.

3.2 The restriction of Φ on $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0$

In this subsection, we will study the restriction of Φ in (5) to the subspace $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0 \subset \mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom}$, i.e., all PL maps from S^k to $\mathcal{Z}_p(X)$ which are homotopic to zero. Note that the image of Φ is in $\left\{\bigoplus_{r\geq k+1, r+s=k+1} H^{p+r,p+s}(X)\right\}^*/H_{2p+k+1}(X,\mathbb{Z})$.

Let $\varphi : S^k \to \mathcal{Z}_p(X)$ be an element in $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0$. Denote by c the total $(2p+k)$-cycle (maybe degenerated) determined by φ. Hence there exists a map $\tilde{\varphi} : D^{k+1} \to \mathcal{Z}_p(X)$ such that $\tilde{\varphi}|_{S^k} = \varphi$ and the associated total $(2p+k+1)$-chain $\tilde{c}$ such that the boundary of $\tilde{c}$ is c, i.e., $\partial\tilde{c} = c$.

The restriction of the generalized Abel-Jacobi map Φ to the subspace of $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0$ is the map

$$\Phi_0 : \mathrm{Map}(S^k, \mathcal{Z}_p(X))_0 \to \left\{\bigoplus_{r\geq k+1, r+s=k+1} H^{p+r,p+s}(X)\right\}^* \Big/ H_{2p+k+1}(X,\mathbb{Z}).$$

Note that

$$\tilde{c} \in \left\{\bigoplus_{r,s\geq 0, r+s=k+1} \mathcal{E}_{p+r,p+s}\right\}$$

and

$$c \in \left\{\bigoplus_{r,s\geq 0, r+s=k} \mathcal{E}_{p+r,p+s}\right\}.$$

Hence

$$\Phi_\varphi(\omega) = \int_{\tilde{c}} \omega = 0$$

for

$$\omega \in \left\{ \bigoplus_{r>k+1, r+s=k+1} \mathcal{E}^{p+r,p+s} \right\} \quad with \quad d\omega = 0$$

by the reason of type. Therefore $\Phi_\varphi = 0$ on

$$\bigoplus_{r>k+1, r+s=k+1} H^{p+r,p+s}(X).$$

That is to say, the image of Φ on the subspace $\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0$ is in

$$H^{p+k+1,p}(X)^* \Big/ \left\{ H^{p+k+1,p}(X)^* \cap \rho(H_{2p+k+1}(X, \mathbb{Z})) \right\}.$$

3.3 The reduction of Φ to $L_p H_{2p+k}(X)_{hom}$

We reduce the domain Φ to the quotient

$$\mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom} / \mathrm{Map}(S^k, \mathcal{Z}_p(X))_0 \cong \pi_0(\mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom}.$$

Now, if there are two PL maps $\varphi : S^k \to \mathcal{Z}_p(X)$ and $\varphi' : S^k \to \mathcal{Z}_p(X)$ such that φ is homotopic to φ'. Denote by c (resp. c') the total $(k+2p)$-cycle determined by φ (resp. φ'). For

$$\omega \in \left\{ \bigoplus_{r>k+1, r+s=k+1} \mathcal{E}^{p+r,p+s} \right\}, \quad d\omega = 0,$$

since $c - c' \overset{hom}{\sim} 0$, we have $\Phi_{\varphi-\varphi'}\omega = \Phi_\varphi\omega - \Phi_{\varphi'}\omega = 0$ and

$$\Phi_\varphi = \Phi_{\varphi'} \in \left\{ \bigoplus_{r>k+1, r+s=k+1} H^{p+r,p+s}(X) \right\}^* \Big/ H_{2p+k+1}(X, \mathbb{Z})$$

by the discuss in Section 3.2.

Therefore, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}(S^k, \mathcal{Z}_p(X))_0 & \overset{i}{\hookrightarrow} & \mathrm{Map}(S^k, \mathcal{Z}_p(X))_{hom} \\
\downarrow \Phi_0 & & \downarrow \Phi \\
H^{p+k+1,p}(X)^*/H_{2p+k+1}(X, \mathbb{Z}) & \overset{i}{\hookrightarrow} & \left\{ \bigoplus_{r \geq k+1, r+s=k+1} H^{p+r,p+s}(X) \right\}^* / H_{2p+k+1}(X, \mathbb{Z}).
\end{array}$$

From this, we reduces Φ to a map

$$\Phi_{tr}:\pi_0(\mathrm{Map}(S^k,\mathcal{Z}_p(X))_{hom}\to\left\{\bigoplus_{r>k+1,r+s=k+1}H^{p+r,p+s}(X)\right\}^*\Big/H_{2p+k+1}(X,\mathbb{Z})\qquad(6)$$

given by $\Phi_{tr}(\varphi)=\Phi_\varphi$. Here $/H_{2p+k+1}(X,\mathbb{Z})$ means modulo the image of the composition map

$$H_{2p+k+1}(X,\mathbb{Z})\xrightarrow{\rho}H_{2p+k+1}(X,\mathbb{C})=\left\{\bigoplus_{r+s=2p+k+1}H^{r,s}(X)\right\}^*\to\left\{\bigoplus_{r>k+1,r+s=k+1}H^{p+r,p+s}(X)\right\}^*.$$

We complete the construction of the generalized Abel-Jacobi map on homologically trivial part in Lawson homology

$$L_pH_{2p+k}(X)_{hom}:=\pi_0(\mathrm{Map}(S^k,\mathcal{Z}_p(X))_{hom},$$

i.e., the kernel of the natural transformation $\Phi_{p,2p+k}:L_pH_{2p+k}(X)\to H_{2p+k}(X,\mathbb{Z})$.

Remark 3.2 *This map (6) defined above is exactly the usual Abel-Jacobi map on Griffiths group when $k=0$ since there is a natural isomorphism $L_pH_{2p}(X)_{hom}\cong\mathrm{Griff}_p(X)$ (cf. [F1]). This map Φ on $L_0H_k(X)_{hom}$ is trivial since $L_0H_k(X)_{hom}=0$ by Dold-Thom theorem (cf. [DT]).*

Remark 3.3 *Our generalized Abel-Jacobi map has been generalized to Lawson homology groups by the author. The range of the more generalized Abel-Jacobi map will be certain Deligne (co)homology. The tools used there are "sparks" and "differential characters" systematically studied by Harvey, Lawson and Zweck [HLZ] and [HL].*

Remark 3.4 *Sometimes we also use $AJ_X(c)$ to denote $\Phi_{tr}(\varphi)$, where c is the cycle determined by φ.*

Remark 3.5 *Prof. M. Walker told me the Suslin Conjecture would imply the existence of such a generalized Abel-Jacobi map, at least for smooth projective 4-folds with $p=1$ and $k=1$ in the equation (6). In his recent paper, Walker has defined a morphic Abel-Jacobi map from algebraically trivial part of p-cycles to p-th morphic Jacobian [Wa].*

4 The non-triviality of the generalized Abel-Jacobi map

The natural question is the existence of smooth projective varieties such that the generalized Abel-Jacobi map Φ_{tr} on $L_pH_{2p+k}(X)_{hom}$ is non-trivial for both $p>0$ and

$k > 0$. The following example is a family of smooth 4-dimensional projective varieties X with $L_1H_3(X)_{hom} \neq 0$, even modulo torsion.

Example: Let E be a smooth elliptic curve and Y be a smooth projective algebraic variety such that the Griffiths group of 1-cycles of Y tensored with $\mathbb{Q}$ is nontrivial. Set $X = E \times Y$. Let $[\omega] \in H^{4,0}(X)$ be a non zero element. By Künnuth formula, we have $[\omega] = [\alpha] \wedge [\beta]$ for some $0 \neq [\alpha] \in H^{1,0}(E)$ and $0 \neq [\beta] \in H^{3,0}(Y)$.

Let $\imath : S^1 \to E$ be a homeomorphism onto its image such that $\imath(S^1) \subset E$ is not homologous to zero in $H_1(E, \mathbb{Z})$. Let $\varphi : S^1 \to \mathcal{Z}_1(X)$ be a continuous map given by

$$\varphi(t) = (\imath(t), W) \in \mathcal{Z}_1(X), \tag{7}$$

where $W \in \mathcal{Z}_1(Y)$ a fixed element such that W is homologous to zero but W is not algebraic equivalent to zero, i.e., $W \in \mathrm{Griff}_1(\mathrm{Y})$. The existence of W is the assumption. Then there exists an integral topological chain U such that $\partial U = W$. Using the notation above, the cycle c determined by φ is $\imath(S^1) \times W$. Now $c = \imath(S^1) \times W$ is homologous to zero in X. Indeed,

$$\partial(\imath(S^1) \times U) = \partial(\imath(S^1)) \times U + \imath(S^1) \times \partial U = \imath(S^1) \times W = c \tag{8}$$

Hence $\partial \tilde{c} = \imath(S^1) \times U + \gamma + \partial(\mathit{something})$, where $\partial\gamma = 0$. Therefore we have

$$\int_{\tilde{c}} \omega = \int_{\imath(S^1)\times U} \omega = \Big(\int_{\imath(S^1)} \alpha\Big) \cdot \Big(\int_U \beta\Big).$$

Proposition 4.1 *Suppose Y is a smooth threefold and $W \in \mathcal{Z}_1(Y)$ such that the image $AJ_Y(W)$ of W under the Griffiths' Abel-Jacobi map AJ_Y is non torsion in $H^{3,0}(Y)^*/\mathrm{Im}H_3(Y, \mathbb{Z})$. The map φ is given by (7) as above. Then the map $\Phi_{tr}(\varphi) \in H^{4,0}(X)/\mathrm{Im}H_4(X, \mathbb{Z})$ is nontrivial, even modulo torsion.*

Proof. By Künneth formula, we have $H^{4,0}(E \times Y) \cong H^{1,0}(E) \otimes H^{3,0}(Y)$ and $H_4(E \times Y, \mathbb{Z}) \cong H_4(Y, \mathbb{Z}) \oplus \{H_1(E, \mathbb{Z}) \otimes H_3(Y, \mathbb{Z})\} \oplus \{H_2(E, \mathbb{Z}) \otimes H_2(Y, \mathbb{Z})\}$ modulo torsion. Let

$$\pi : H_4(E \times Y, \mathbb{Z}) \to \{H^{4,0}(E \times Y)\}^*$$

be the natural map given by $\pi(u)(\alpha \otimes \beta) = \int_u \alpha \wedge \beta$ for $u \in H_4(E \times Y, \mathbb{Z})$ and $\alpha \in H^{1,0}(E)$ and $\beta \in H^{3,0}(Y)$. Now $\pi(u) \neq 0$ only if $u \in H_1(E, \mathbb{Z}) \otimes H_3(Y, \mathbb{Z})$. Hence we get

$$\{H^{4,0}(E \times Y)\}^*/\mathrm{Im}H_4(E \times Y, \mathbb{Z}) \cong \{H^{1,0}(E)^* \otimes H^{3,0}(Y)^*\}/\mathrm{Im}\{H_1(E, \mathbb{Z}) \otimes H_3(Y, \mathbb{Z})\}.$$

Therefore, by the definition of generalized Abel-Jacobi map and (8), we have

$$AJ_Y(\imath(S^1) \times W)(\alpha \wedge \beta) = \Phi_{tr}(\varphi)(\alpha \wedge \beta) = \Big(\int_{\imath(S^1)} \alpha\Big) \cdot \Big(\int_U \beta\Big) = \Big(\int_{\imath(S^1)} \alpha\Big) \cdot (AJ_Y(W)(\beta))$$

i.e., $AJ_Y(\imath(S^1)\times W)=\int_{\imath(S^1)}\otimes AJ_Y(W)$.

Note that the map $\int_{\imath(S^1)}: H^{1,0}(E)\to\mathbb{C}$ is in the image of the embedding $H_1(E,\mathbb{Z})\hookrightarrow H^{1,0}(E)^*$. But $AJ_Y(W)$ is a non-torsion element in $H^{3,0}(Y)^*/\mathrm{Im}H_3(Y,\mathbb{Z})$. Now the conclusion of the proposition is from the following lemma.

Lemma 4.1 *Let V_m and V_n be two $\mathbb{C}$-vector spaces of dimension of m and n, respectively. Suppose that $\Lambda_m\subset V_m$, $\Lambda_n\subset V_n$ be two lattices, respectively. If $b\in V_n$ is a non torsion element in V_n, i.e., kb is not zero in Λ_n for any $k\in\mathbb{Z}^*$, then $a\otimes b$ is not in $\Lambda_m\otimes\Lambda_n$ for any $0\neq a\in\Lambda_m$.*

Proof. Set $\mathrm{rank}(\Lambda_m)=m_0$, $\mathrm{rank}(\Lambda_n)=n_0$. Let $\{e_i\}_{i=1}^{m_0}$, $\{f_j\}_{j=1}^{n_0}$ be two integral basis of Λ_m, Λ_n, respectively. If the conclusion in the lemma fails, then

$$a\otimes b=\sum_{i=1}^{m_0}\sum_{j=1}^{n_0}k_{ij}e_i\otimes f_j, \tag{9}$$

for some $k_{ij}\in\mathbb{Z}$. By taking the conjugation, we can suppose that V_m and V_n are real vector spaces with lattices Λ_m and Λ_n, respectively.

Suppose that $a=\sum_{i=1}^{m_0}k_ie_i$, where $k_i\in\mathbb{Z}$, $i=1,\cdots,m_0$ are not all zeros. The equation (9) reads as

$$\sum_1^{m_0}k_ie_i\otimes b=\sum_{i=1}^{m_0}\sum_{j=1}^{n_0}k_{ij}e_i\otimes f_j$$

i.e.,

$$\sum_{i=1}^{m_0}e_i\otimes\big(k_ib-\sum_{j=1}^{n_0}k_{ij}f_j\big)=0.$$

Since $\{e_i\}_{i=1}^{m_0}$ is a basis in Λ_m and hence they are linearly independent over $\mathbb{R}$ in V_m, we get

$$k_ib-\sum_{j=1}^{n_0}k_{ij}f_j=0 \tag{10}$$

for any $i=1,2,\cdots,m_0$. By assumption, at least one of k_i is nonzero since a is nonzero vector in V_m. The equation (10) contracts to the assumption that kb is not in Λ_n for any $k\in\mathbb{Z}^*$. This completes the proof of the lemma and hence the proof of the proposition. □

□

More generally, we have the following Proposition

Proposition 4.2 *Suppose Y is a smooth threefold such that the image $AJ_Y(\mathrm{Griff_1(Y)})$ of* $\mathrm{Griff_1(Y)}$ *under the Griffiths' Abel-Jacobi map AJ_Y tensored by $\mathbb{Q}$ is are infinitely dimensional $\mathbb{Q}$-vector space over $\mathbb{Q}$ in $\{H^{3,0}(Y)^*/\mathrm{Im}H_3(Y,\mathbb{Z})\}\otimes\mathbb{Q}$. For each $W\in\mathrm{Griff_1(Y)}$, The map φ_W is given by (7) as above. Then the image*

$$\left\{\Phi_{tr}(\varphi_W)\middle|W\in\mathrm{Griff_1(Y)}\right\}\otimes\mathbb{Q}\subset\left\{H^{4,0}(X)/\mathrm{Im}H_4(X,\mathbb{Z})\right\}\otimes\mathbb{Q}$$

is an infinite dimensional $\mathbb{Q}$-vector space.

Proof. We only need to show that:

($*$) Let $N>0$ be an integer and $W_1,\cdots,W_N\in\mathrm{Griff}_1(Y)$ be N linearly independent elements under Griffiths Abel-Jacobi map. Then $\varphi_{W_1},\cdots,\varphi_{W_N}\in L_1H_3(E\times Y)_{hom}\otimes\mathbb{Q}$ are linearly independent even under the generalized Abel-Jacobi map.

The claim ($*$) follows easily from Proposition 4.1 above since if $\varphi_{W_1},\cdots,\varphi_{W_N}$ are linearly dependent implies that $W_1,\cdots,W_N$ are linearly dependent by Proposition 4.1. This contradicts to the assumption.

□

Now for suitable choice of the 3-dimensional projective Y, for example, the general quintic hypersurface in P^4 (cf. [G]) or the Jacobian of a general algebraic curve with genus 3 (cf. [C]) and the 1-cycle W whose image under Abel-Jacobi map is nonzero, in fact, it is infinitely generated for general quintic hypersurface in P^4 (cf. [Cl]). Recall the definition of Abel-Jacobi map, $AJ_Y(W)=\int_U$ module lattice $H^3(Y,\mathbb{Z})$, we have $\int_U\beta\neq0$ for this choice of W and some nonzero $[\beta]\in H^{3,0}(Y)$.

This example also gives an affirmative answer the following question:

Question: Can one show that $L_pH_{2p+j}(X)_{hom}$ is nontrivial or even infinitely generated for some projective variety X where $j>0$?

Remark 4.1 *From the proof of the above propositions, we see that the non-triviality of Griffiths' Abel-Jacobi map on Y implies the non-triviality of the generalized Abel-Jacobi map on homologically trivial part of certain Lawson homology groups for X, i.e., all the Abel-Jacobi invariants can be found by generalized Abel-Jacobi map. In [Cl], Clemens showed the for general quintic 3-fold, the image of the Griffiths group under the Griffiths' Abel-Jacobi map is infinitely generated, even modulo torsion.*

Remark 4.2 *Friedlander proved in [F2] the non-triviality of $L_rH_{2p}(X)_{hom}$ for certain complete intersections by using Nori's method in [N], which is totally different the construction here. There is no claim of any kind of infinite generated property of Lawson homology in his paper.*

Remark 4.3 *Nori [N] has generalized Theorem 0.1 and has shown that even the Griffiths' Abel-Jacobi map is trivial on some Griffiths group but the Griffiths group itself is nontrivial, even non torsion. By using a total different, explicit and elementary construction, the author has constructed singular rational 4-dimensional projective varieties X such that $L_1H_3(X)_{hom}$ is infinitely generated [H]. But the Able-Jacobi map is not defined on singular projective variety (at least I don't know).*

From the proof of Proposition 4.1, we observe that, for Y as above, and M is a projective manifold, if there is a map $i: S^k \to M$ such that

$$\int_{i(S^k)} : H^{k,0}(M) \to \mathbb{C}$$

is non-trivial as element in $\{H^{k,0}(M)\}^*$, then the value of the generalized Abel-Jacobi map Φ_{tr} at $\varphi : S^k \to \mathcal{Z}_1(X)$ defined by

$$\varphi(t) = (i(t), W) \in \mathcal{Z}_1(M \times Y)$$

is non-trivial, even modulo torsion.

Note that if the *complex Hurewicz homomorphism* $\rho \otimes \mathbb{C} : \pi_k(X) \otimes \mathbb{C} \to H_k(M, \mathbb{C})$ is surjective or even a little weaker condition, i.e., the composition

$$\pi_k(X) \otimes \mathbb{C} \to H_k(M, \mathbb{C}) \to \{H^{k,0}(M)\}^*$$

is surjective, we have the non-triviality of the map $\int_{i(S^k)} : H^{k,0}(M) \to \mathbb{C}$ if $H^{k,0}(M) \neq 0$. Here the map $\pi : H_k(M, \mathbb{C}) \to \{H^{k,0}(M)\}^*$ is the Poincaré duality the projection $H^k(M, \mathbb{C}) \to H^{k,0}(M)$ in Hodge decomposition.

As a direct application to the Main Theorem in [[DGMS], §6] and also Theorem 14 in [NT], we have the following result on higher dimensional hypersurface.

Proposition 4.3 *Let M be a smooth hypersurface in P^{n+1} and $n > 1$. Then the composition map*

$$\pi_k(X) \otimes \mathbb{C} \to H_k(M, \mathbb{C}) \to \{H^{k,0}(M)\}^*$$

is surjective for any simply connected Kähler manifolds.

Therefore we obtain the following result:

Theorem 4.1 *For any $k \geq 0$, there exist a projective manifold X of dimension $k+3$ such that $L_1H_{k+2}(X)_{hom} \otimes \mathbb{Q}$ is* nontrivial *or even* infinite *dimensional over $\mathbb{Q}$.* □

By using the Projective Bundle Theorem in [FG], we get the following result:

Theorem 4.2 *For any $p > 0$ and $k \geq 0$, there is a smooth projective variety X such that $L_pH_{k+2p}(X)_{hom} \otimes \mathbb{Q}$ is* infinite *dimensional vector space over $\mathbb{Q}$.* □

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References

[C] G. Ceresa, *C is not algebraically equivalent to C^- in its Jacobian.* Ann. of Math. (2) 117 (1983), no. 2, 285–291.

[Cl] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated.* Inst. Hautes études Sci. Publ. Math. No. 58, (1983), 19–38 (1984).

[DGMS] P. Deligne; P. Griffiths; J. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds.* Invent. Math. 29 (1975), no. 3, 245–274.

[DT] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte.* (German) Ann. of Math. (2) 67 1958 239–281.

[F1] E. Friedlander, *Algebraic cycles, Chow varieties, and Lawson homology.* Compositio Math. 77 (1991), no. 1, 55–93.

[F2] E. Friedlander, *Relative Chow correspondences and the Griffiths group.* (English. English, French summary) Ann. Inst. Fourier (Grenoble) 50 (2000), no. 4, 1073–1098.

[FL1] E. Friedlander and B. H. Lawson, *A theory of algebraic cocycles.* Ann. of Math. (2) 136 (1992), no. 2, 361–428.

[FL2] E. Friedlander and B. H. Lawson, *Graph mappings and poincaré duality*, preprint.

[FG] E. Friedlander and O. Gabber, *Cycle spaces and intersection theory. Topological methods in modern mathematics* (Stony Brook, NY, 1991), 325–370, Publish or Perish, Houston, TX, 1993.

[FM] E. Friedlander and B. Mazur, *Filtrations on the homology of algebraic varieties. With an appendix by Daniel Quillen.* Mem. Amer. Math. Soc. 110 (1994), no. 529, x+110 pp.

[G] P. Griffiths, *On the periods of certain rational integrals I, II.* Ann. of Math. (2) 90(1969), 460-495; ibid. (2) 90(1969) 496–541.

[HL] R. Harvey; B. Lawson, *D-bar sparks, I.* arXiv.org:math.DG/0512247

[HLZ] R. Harvey; B. Lawson and J. Zweck, *The de Rham-Federer theory of differential characters and character duality.* Amer. J. Math. 125 (2003), no. 4, 791–847

[Hi] H. Hironaka, Triangulations of algebraic sets, Proc. of Symposia in Pure Math 29 (1975), 165-185.

[H] W. Hu, *Infinitely generated Lawson homology groups on some rational projective varieties.* arXiv.org:math.AG/0602517

[L1] B. Lawson, *Algebraic cycles and homotopy theory.* Ann. of Math. **129**(1989), 253-291.

[L2] B. Lawson, *Spaces of algebraic cycles.* pp. 137-213 in Surveys in Differential Geometry, 1995 vol.2, International Press, 1995.

[NT] J. Neisendorfer and L. Taylor, *Dolbeault homotopy theory.* Trans. Amer. Math. Soc. 245 (1978), 183–210.

[N] M. Nori, *Algebraic cycles and Hodge-theoretic connectivity.* Invent. Math. 111 (1993), no. 2, 349–373.

[V1] C. Voisin, *Hodge theory and complex algebraic geometry. I.* Translated from the French original by Leila Schneps. Cambridge Studies in Advanced Mathematics, 76. Cambridge University Press, Cambridge, 2002. x+322 pp. ISBN 0-521-80260-1

[Wa] M. Walker, *The morphic Abel-Jacobi map.* `http://www.math.uiuc.edu/K-theory/740/`

[W] G. W. Whitehead, *Elements of homotopy theory.* Graduate Texts in Mathematics, 61. Springer-Verlag, New York-Berlin, 1978. xxi+744 pp. ISBN 0-387-90336-4

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